ON SOME ESTIMATES FOR PROJECTION OPERATOR IN BANACH SPACE

Ya. I. Alber* **and A. I. Notik**
Department of Mathematics
Technioin-Israel Institute of Technology
Haifa 32000, Israel

Abstract - Previously unknown estimates of uniform continuity of projection operators in Banach space have been obtained. They can be used in the investigations of approximation methods, in particular, the method of quasisolutions, methods of regularization and penalty functions, for solving nonlinear problems on exact and peturbed sets (see [1, 2]).

1 Introduction

Metric projection operators on convex closed sets in Hilbert and Banach spaces are widely used in functional and numerical analysis, theory of optimization and theory of approximation. Terefore their properties are actively studying .

Metric projection operators can be defined in a same way in Hilbert and Banach spaces.

Let B be a real, uniformly convex and uniformly smooth (reflexive) Banach space with B^* its dual space [8] and $J : B \to B^*$ a normalized duality mapping, Ω a closed convex set in B, $< w, v >$ a dual product in B, i.e. pairing between $w \in B^*$ and $v \in B$ ($(\cdot, \cdot)$ is inner product in Hilbert space); the sigs $||\cdot||$, $||\cdot||_{B^*}$ and $||\cdot||_H$ denote the norms in Banach space B, Banach space B^* and Hilbert space H, respectively. Let $P_\Omega x$ and $P_\Omega y$ be projections of elements x and y onto Ω in the sense of best approximation:

Definition 1.1 *The operator $P_\Omega : B \to \Omega \subset B$ is called to be metric projection operator in Banach space B if it gives the correspondence between an arbitrary point $x \in B$ and nearest point $\bar{x} \in \Omega$ according to minimization problem*

$$P_\Omega x = \bar{x}; \quad \bar{x} : ||x - \bar{x}|| = \inf_{\xi \in \Omega} ||x - \xi||. \tag{1.1}$$

*This research was supported in part by the Ministry of Science Grant 3481-1-91 and by the Ministry of Absorption Center for Absorption in Science.

It is well known that the metric projection operator in Hilbert space H is:
1) monotone (accretive), i.e. for all $x, y \in H$

$$(\bar{x} - \bar{y}, x - y) \geq 0,$$

2) nonexpansive, i.e. it satisfies the Lipschitz condition (with the constant 1)

$$||\bar{x} - \bar{y}||_H \leq ||x - y||_H, \qquad \forall x, y \in H.$$

3) It gives an absolutely best approximation for arbitrary elements from Hilbert space by the elements of convex closed set Ω , i.e.

$$||\bar{x} - \xi||_H^2 \leq ||x - \xi||_H^2 - ||x - \bar{x}||_H^2, \qquad \forall \xi \in \Omega,$$

4) The operator P_Ω satisfies the following necessary and sufficient condition

$$(x - \bar{x}, \bar{x} - \xi) \geq 0, \qquad \forall \xi \in \Omega.$$

Mainly, these properties lead to a variety of applications of the metric projection operator in Hilbert spaces.

Metric projection operators in Banach spaces do not have the properties 1)-3) mentioned above . However, the property 4) for Banach spaces has been established in [9] (see [7] also) in a form

$$< J(x - \bar{x}), \bar{x} - \xi > \geq 0, \qquad \forall \xi \in \Omega. \tag{1.2}$$

This property is often used in applications. A question connected with smoothness of the metric projection operators in Banach spaces is not of less importance either. It is known that in uniformly smooth Banach space the operator P_Ω is always continuous but not always uniformly continuous. In the present paper, we prove the uniform continuty of P_Ω with respect to changes of the argument x (Theorem 3.1) in uniformly smooth and uniformly convex Banach spaces and the uniform continuty of P_Ω with respect to changes of set Ω (Theorem 3.6) in uniformly convex Banach spaces. This properties have been applied for investigation of stability of approximation methods for solving equations and variational inequalities in Banach spaces [1, 2].

2 Auxiliary lemmas

In the prooving main results, we use some properties of duality mappings which were in most of known monographs devoted to the theory of monotone and accretive operators.
i) The normalized duality mapping J is monotone operator, i.e.

$$< Jx - Jy, x - y > \geq 0, \qquad \forall \xi \in \Omega.$$

ii) In uniformly convex Banach spaces, mapping J is uniformly monotone operator on each bouded set, i.e. for every $R > 0$ there exists a function $\psi_R(t) > 0, \psi_R(0) = 0$, such that for all x and y in B, $||x|| \leq R$ and $||y|| \leq R$,

$$< Jx - Jy, x - y > \geq \psi_R(||x - y||).$$

iii) In uniformly smooth Banach spaces, mapping J is uniformly continuous operator on each bouded set, i.e. for every $R > 0$ there exists a continuous function $\omega_R(t) > 0$, $\omega_R(0) = 0$, such that for all $||x|| \leq R$ and $||y|| \leq R$,

$$||Jx - Jy||_{B^*} \leq \omega_R(||x - y||).$$

However, during a big period of time the functions $\psi_R(t)$ and $\omega_R(t)$ had no quantitative description. And only in 1984, such estimates have been obtained in [3] (see also [4]). But in [3, 4] we did not proved their detailed proofs. In this Section we are intended to fill this gap.

Lemma 2.1 *(cf.[3, 4]). Let B be an uniformly convex Banach spaces. If $\delta_B(\epsilon)$ is modulus of the convexity of space B [8] then the following estimate*

$$< Jx - Jy, x - y > \geq (2L)^{-1}\delta_B(||x - y||/C_1),$$

$$C_1 = 2\max\{1, \sqrt{(||x||^2 + ||y||^2)/2}\}$$

is valid. Here L is the constant from the Figiel's iniquality (2.3), $1 < L < 3,18$.

Proof. In [8] it was shown that if $x \in B, y \in B$ and $||x||^2 + ||y||^2 = 2$, then

$$||(x + y)/2||^2 \leq 1 - \delta_B(||x - y||/2)$$

where $\delta_B(\epsilon)$ is modulus of convexity of space B. It is known that in uniformly convex Banach space, $\delta_B(\epsilon)$ is convex strictly increasing function, $0 \leq \delta_B(\epsilon) < 1, \delta_B(0) = 0$. We denote $R_1^2 = 2^{-1}(||x||^2 + ||y||^2)$ (x and y are not zero at the same time) and introduce new variables $\tilde{x} = x/R_1$ and $\tilde{y} = y/R_1$. Then

$$||\tilde{x}||^2 + ||\tilde{y}||^2 = R_1^{-2}(||x||^2 + ||y||^2) = 2.$$

Therefor, for $\tilde{x}$ and $\tilde{y}$ the inequality

$$||(\tilde{x} + \tilde{y})/2||^2 \leq 1 - \delta_B(||\tilde{x} - \tilde{y}||/2)$$

is valid.

Let us return to the old variables x and y. We obtain

$$||(x + y)/2R_1||^2 \leq 1 - \delta_B(||x - y||/2R_1)$$

or

$$||(x + y)/2||^2 \leq (||x||^2 + ||y||^2)/2 - R_1^2\delta_B(||x - y||/2R_1), \tag{2.1}$$

$$R_1 = \sqrt{(||x||^2 + ||y||^2)/2}.$$

Consider two cases:

1. $R_1 \geq 1$. Then

$$||(x+y)/2||^2 \leq (||x||^2 + ||y||^2)/2 - \delta_B(||x-y||/2R_1). \tag{2.2}$$

2. $R_1 \leq 1$. From (2.1) and the inequality (see [6])

$$\epsilon^2 \delta_B(\eta) \geq (4L)^{-1} \eta^2 \delta_B(\epsilon), \quad \forall \eta \geq \epsilon > 0 \tag{2.3}$$

we have

$$\delta_B(||x-y||/2R_1) \geq R_1^{-2}(4L)^{-1}\delta_B(||x-y||/2)$$

and

$$||(x+y)/2||^2 \leq (||x||^2 + ||y||^2)/2 - (4L)^{-1}\delta_B(||x-y||/2). \tag{2.4}$$

In vertue of $\min\{1, (4L)^{-1}\} = (4L)^{-1}$, (2.2) and (2.4) if joined together give

$$||(x+y)/2||^2 \leq (||x||^2 + ||y||^2)/2 - (4L)^{-1}\delta_B(||x-y||/C_1), \;\; C_1 = 2\max\{1, R_1\}.$$

If $||x|| \leq R$ and $||y|| \leq R$, then $C_1 = 2\max\{2, R\}$ is the absolute constant.

Let $\varphi(x) = ||x||^2/2$. Then

$$\varphi((x+y)/2) \leq 2^{-1}\varphi(x) + 2^{-1}\varphi(y) - (8L)^{-1}\delta_B(||x-y||/C_1). \tag{2.5}$$

In [12] it was proven that if a convex functional $\phi(x)$ difined on convex closed set Ω satisfies the inequality

$$\phi(\frac{1}{2}x + \frac{1}{2}y) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y) - \kappa(||x-y||),$$

where $\kappa(t) \geq 0, \kappa(t_0) > 0$ for some $t_0 > 0$, then $\phi(x)$ is uniformly convex functional with modulus of convexity $\delta(t) = 2\kappa(t)$, and

$$\phi(x) \geq \phi(y) + < l(y), x-y > + 2\kappa(||x-y||),$$

$$< l(x) - l(y), x-y > \geq 4\kappa(||x-y||) \tag{2.6}$$

for all $l(x) \in \partial\phi(x)$. Here $\partial\phi(x)$ is the set of all support functionals (the set of all subgradients) of $\phi(x)$ at the point $x \in \Omega$.

The formula (2.5) then shows that the functional $\varphi(x)$ is uniformly convex with modulus of convexity

$$\delta(||x-y||) = (4L)^{-1}\delta_B(||x-y||/C_1).$$

Let us apply now (2.6). Because $\varphi(x)$ is differentiable functional and $\varphi'(x) = grad(||x||^2/2) = Jx$, then

$$< Jx - Jy, x-y > \geq (2L)^{-1}\delta_B(||x-y||/C_1). \tag{2.7}$$

Lemma is proved.

Lemma 2.2 *(cf.[3, 4]). Let B^* be an uniformly convex Banach spaces. If $\delta_{B^*}(\epsilon)$ is modulus of the convexity of space $_{B^*}$ and $g_{B^*}(\epsilon) = \delta_{B^*}(\epsilon)/\epsilon$, $g_{B^*}^{-1}(\cdot)$ is an inverse function, then the following estimate*

$$||Jx - Jy||_{B^*} \leq C_1(g_{B^*}^{-1}(2C_1L||x-y||))$$

. is valid. Here L is the constant from Lemma 2.1.

Proof. Analogously to (2.7), we can obtain the inequality

$$< Jx - Jy, x - y >\geq (2L)^{-1}\delta_{B^*}(||Jx - Jy||_{B^*}/C_2). \tag{2.8}$$

for uniformly smooth space B. From (2.8) one has

$$||Jx - Jy||_{B^*}||x - y|| \geq (2L)^{-1}\delta_{B^*}(||Jx - Jy||_{B^*}/C_2)$$

Since $g_{B^*}(\epsilon) = \delta_{B^*}(\epsilon)/\epsilon$, we can write

$$g_{B^*}(||Jx - Jy||_{B^*}/C_2) \leq 2C_2L||x - y||. \tag{2.9}$$

This proves Lemma.

3 Main results

The first problem. Estimate $||P_\Omega x - P_\Omega y||$ via $||x - y||$.

Theorem 3.1 *Let B be the uniformly convex and uniformly smooth Banach spaces. If $\delta_B(\epsilon)$ is modulus of the convexity of space B and $g_B(\epsilon) = \delta_B(\epsilon)/\epsilon, g_B^{-1}(\cdot)$ is an inverse function, then*

$$||P_\Omega x - P_\Omega y|| \leq Cg_B^{-1}(2LC^2g_{B^*}^{-1}(2CL||x - y||)) \tag{3.1}$$

where L is the constant from Lemma 2.1, and

$$C = 2\max\{1,\ ||x - P_\Omega y||,\ ||y - P_\Omega x||\}.$$

Proof. Let us denote $\bar{x} = P_\Omega x$ and $\bar{y} = P_\Omega y$. It is known from (1.2) that

$$< J(x - \bar{x}), \bar{x} - \bar{y} > \geq\ 0,\ \ < J(y - \bar{y}), \bar{y} - \bar{x} > \geq\ 0.$$

Therefore

$$< J(x - \bar{y}), \bar{x} - \bar{y} > \geq < J(x - \bar{x}) - J(x - \bar{y}), \bar{y} - \bar{x} > . \tag{3.2}$$

Using Lemma 2.1 we obtain

$$< J(x - \bar{x}) - J(x - \bar{y}), \bar{y} - \bar{x} > \geq\ (2L)^{-1}\delta_B(||\bar{x} - \bar{y}||/C_1). \tag{3.3}$$

Here

$$C_1 = 2\max\{1, \sqrt{(||x-\bar{x}||^2 + ||x-\bar{y}||^2)/2}\}.$$

Now

$$< J(x-\bar{y}) - J(y-\bar{y}), \bar{x}-\bar{y} > \geq \ (2L)^{-1}\delta_B(||\bar{x}-\bar{y}||/C_1)$$

follows from (3.2) and (3.3). Applying the Caushy-Schwarz inequality we get

$$g_B(||\bar{x}-\bar{y}||/C_1) \leq 2LC_1||J(x-\bar{y}) - J(y-\bar{y})||_{B^*}. \tag{3.4}$$

In Lemma 2.2 the estimate

$$||J(x-\bar{y}) - J(y-\bar{y})||_{B^*} \leq C_2(g_{B^*}^{-1}(2C_2L||x-y||)). \tag{3.5}$$

was obtained, where

$$C_2 = 2\max\{1, \sqrt{(||x-\bar{y}||^2 + ||y-\bar{y}||^2)/2}\} \leq$$

$$2\max\{1, ||x-\bar{y}||, ||y-\bar{y}||\} \leq 2\max\{1, ||x-\bar{y}||, ||y-\bar{x}||\}.$$

Thus, the inequality

$$||P_\Omega x - P_\Omega y|| \leq C_1 g_B^{-1}(2LC_1C_2 g_{B^*}^{-1}(2C_2L||x-y||))$$

is realized from (3.4) and (3.5). Finally, we get (3.1) because $C_1 \leq 2\max\{1, ||x-\bar{y}||\}$.

Remark 3.2 *It follows from (3.1) that the projection operator P_Ω is uniformly continuous on every bounded set of Banach space B.*

Remark 3.3 *If B and B^* are Hilbert spaces H, then the proof of Theorem 3.1 gives $||P_\Omega x - P_\Omega y||_H \leq ||x-y||_H$. It is useful to remind that $\epsilon^2/8 \leq \delta_H(\epsilon) \leq \epsilon^2/4$.*

Remark 3.4 *If $y \in \Omega$ then in (3.1) $C = 2\max\{1, 2||x - P_\Omega y||\}$; if $x \in \Omega$, then $C = 2\max\{1, 2||y - P_\Omega x||\}$, i.e. $C = 2\max\{1, 2||x-y||\}$.*

Remark 3.5 *Instead of (3.5) one can use the estimates of duality mapping with gauge function too [11, 13].*

Let Ω_1 and Ω_2 be convex closed sets, $x \in B$ and $H(\Omega_1, \Omega_2) \leq \sigma$, where

$$H(\Omega_1, \Omega_2) = \max\{\sup_{z_1\in\Omega_1}\inf_{z_2\in\Omega_2}||z_1-z_2||, \quad \sup_{z_1\in\Omega_2}\inf_{z_2\in\Omega_1}||z_1-z_2||\}$$

is a Hausdorff distance between Ω_1 and Ω_2.

The second problem. Estimate $||P_{\Omega_1}x - P_{\Omega_2}x||$ via σ.

Theorem 3.6 *If B is a uniformly convex space, $\delta_B(\epsilon)$ is modulus of the convexity, and $\delta_B^{-1}(\cdot)$ is an inverse function, then*

$$||P_{\Omega_1}x - P_{\Omega_2}x||_B \leq C_1\delta_B^{-1}(4L(d+r)\sigma), \tag{3.6}$$

where L is the constant from Lemma 2.1, $r = ||x||$, $d = \max\{d_1, d_2\}$, $d_i = dist$ (θ, Ω_i), $i = 1, 2$, θ is an origin of space B, $C_1 = 2\max\{1, r+d\}$.

Proof. Denote $\bar{x}_1 = P_{\Omega_1}x$, $\bar{x}_2 = P_{\Omega_2}x$. From Lemma 2.1 we have

$$< J(x-\bar{x}_1) - J(x-\bar{x}_2), \bar{x}_2 - \bar{x}_1 > \geq (2L)^{-1}\delta_B(||\bar{x}_1 - \bar{x}_2||/C), \tag{3.7}$$

$$C = 2\max\{1, ||x-\bar{x}_1||, ||x-\bar{x}_2||\}.$$

There exist $\xi_1 \in \Omega_1$ such that $||\bar{x}_2 - \xi_1|| \leq \sigma$ and

$$< J(x-\bar{x}_1), \bar{x}_2 - \bar{x}_1 > = < J(x-\bar{x}_1), \bar{x}_2 - \xi_1 > + < J(x-\bar{x}_1), \xi_1 - \bar{x}_1 > \leq \sigma||x-\bar{x}_1||,$$

because $H(\Omega_1, \Omega_2) \leq \sigma$, and $< J(x-\bar{x}_1), \xi_1 - \bar{x}_1 > \leq 0$.

In the same way there exist $\xi_2 \in \Omega_2$ such that $||\bar{x}_1 - \xi_2|| \leq \sigma$ and

$$< J(x-\bar{x}_2), \bar{x}_1 - \bar{x}_2 > = < J(x-\bar{x}_2), \bar{x}_1 - \xi_2 > +$$

$$< J(x-\bar{x}_2), \xi_2 - \bar{x}_2 > \leq \sigma||x-\bar{x}_2||,$$

because $H(\Omega_1, \Omega_2) \leq \sigma$, and $< J(x-\bar{x}_2), \xi_2 - \bar{x}_2 > \leq 0$. Therefore

$$< J(x-\bar{x}_1) - J(x-\bar{x}_2), \bar{x}_2 - \bar{x}_1 > \leq \sigma(||x-\bar{x}_1|| + ||x-\bar{x}_2||) \tag{3.8}$$

holds. It is obvious that

$$||x-\bar{x}_1|| \leq ||x - P_{\Omega_1}\theta|| \leq ||x|| + ||P_{\Omega_1}\theta|| \leq r+d,$$

$$||x-\bar{x}_2|| \leq ||x - P_{\Omega_2}\theta|| \leq ||x|| + ||P_{\Omega_2}\theta|| \leq r+d.$$

From (3.7) and (3.8) we can write

$$(2L)^{-1}\delta_B(||\bar{x}_1 - \bar{x}_2||/C_1) \leq 2\sigma(r+d), \; C_1 = 2\max\{1, r+d\}.$$

Hence the estimate (3.6) is valid.

Remark 3.7 *The more exact estimate (3.6) is*

$$||P_{\Omega_1}x - P_{\Omega_2}x||_B \leq C_1\delta_B^{-1}(4LC_2\sigma),$$

$$C_1 = 2\max\{1, ||x-\bar{x}_1||, ||x-\bar{x}_2||\}, \;\; C_2 = 2\max\{||x-\bar{x}_1||, ||x-\bar{x}_2||\}.$$

Let us notice for Hilbert space, that the estimate

$$||P_{\Omega_1}x - P_{\Omega_2}x||_H \le \sqrt{4\sigma(2r+d)+\sigma^2}$$

has been established in [5] (see also [10]). However, applying the proof of Theorem 3.6 for this special case, one can obtain inequality

$$||P_{\Omega_1}x - P_{\Omega_2}x||_H \le \sqrt{2\sigma(r+d)}. \tag{3.9}$$

Indeed,

$$||\bar{x}_1 - \bar{x}_2||_H^2 = ((x-\bar{x}_1)-(x-\bar{x}_2), \bar{x}_2 - \bar{x}_1) \ \le 2\sigma(r+d)$$

takes place. As one can see, the estimate (3.9) is obtained much more easily and of a higher quality, than in [5] and [10].

Let Ω_1 and Ω_2 be convex closed sets, $x \in B, y \in B, H(\Omega_1, \Omega_2) \le \sigma$.

The third problem. Estimate $||P_{\Omega_1}x - P_{\Omega_2}y||$ via $||x-y||$ and σ.

The result follows immediately from (3.1) and (3.6):

$$||P_{\Omega_1}x - P_{\Omega_2}y|| \le Cg_B^{-1}(2LC^2 g_{B^*}^{-1}(2CL||x-y||)) + C_1\delta_B^{-1}(4L(d+r)\sigma),$$

where $r = ||y||, C = 2\max\{1,\ ||x - P_{\Omega_1}y||,\ ||y - P_{\Omega_1}x||\}$, and C_1 and d are determined in Theorem 3.6.

References

[1] Ya.I. Alber, The regularization method for variational inequalities with nonsmooth unbounded operator in Banach space, *Appl. Math. Letters,* **6**, 63-68 (1993).

[2] Ya.I. Alber, On penalty method for variational inequalities with nonsmooth unbounded operator in Banach space (to appear).

[3] Ya.I. Alber, A.I. Notik, Geometric properties of Banach spaces a approximate methods for solving nonlinear operator equations, *Soviet Math. Dokl.* **29**, 611-615 (1984).

[4] Ya.I. Alber, A.I. Notik, Parallelogram inequalities in Banach spaces and some properties of a duality mapping, *Ukrainian Math. J.* **40**, 650-652 (1988).

[5] P. Doktor, M. Kucera, Perturbations of variational inequalities and rate of convergence of solutions, *Chechoslovak. Math. J.* **30**, 426-437 (1980).

[6] T. Figiel, On moduli of convexity and smoothness, *Stud. Math.,* **56** No.2, 121-155 (1976).

[7] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry and nonexpansive mapping,* New York, N.Y., M.Dekker, 1984.

[8] J. Lindenstrauss and L. Tzafriri, *Classical Banach space* , Berlin - Heidelberg - New-York, Springer Verlag, Vol. II, 1979.

[9] J.L. Lions, *Quelques methodes de résolution des problemes aux limites non linéaires* , Dunod Gauthier-Villars, Paris, 1969.

[10] O.A. Liskovets, Ill-posed problems and stability of quasisolutions, *Siberian Math. J.* **10**, 266-274 (1969).

[11] A.I. Notik, Properties of a duality mapping with a scale function, *Soviet Math.* **29**, No.12, 96-98 (1985).

[12] A.A. Vladimirov, Yu.E. Nesterov and Yu.N. Chekanov, Uniformly convex functionals, *Vestnik Moscov. Univ. Ser.15, Vychisl. Mat. Kibernet.,* 1978, no.3, 12-23.

[13] Zong-Ben Xu and G.F. Roach, Characteristic inequalities of uniformly convex and uniform smooth Banach spaces, *J. Math. Anal. Appl.* **157**, 189-210 (1991).